Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Approximation Theory 137 (2005) 22–41

Journal of Approximation Theory

www.elsevier.com/locate/jat

The distance to the functions with range in a given set in Banach spaces of vector-valued continuous functions

Laura Burlando

Dipartimento di Matematica dell'Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy

Received 12 July 2004; accepted 7 July 2005

Communicated by E.W. Cheney
Available online 12 October 2005

Abstract

In this paper we give a formula for the distance from an element f of the Banach space $C(\Omega, X)$—where X is a Banach space and Ω is a compact topological space—to the subset $C(\Omega, S)$ of all functions whose range is contained in a given nonempty subset S of X. This formula is given in terms of the norm in $C(\Omega)$ of the distance function to S that is induced by f (namely, of the scalar-valued function d_f^S which maps $t \in \Omega$ into the distance from $f(t)$ to S), and generalizes the known property that the distance from f to $C(\Omega, V)$ be equal to the norm of d_f^V in $C(\Omega)$ for every vector subspace V of X [Buck, Pacific J. Math. 53 (1974) 85–94, Theorem 2; Franchetti and Cheney, Boll. Un. Mat. Ital. B (5) 18 (1981) 1003–1015, Lemma 2]. Indeed, we prove that the distance from f to $C(\Omega, S)$ is larger than or equal to the norm of d_f^S in $C(\Omega)$ for every nonempty subset S of X, and coincides with it if S is convex or a certain quotient topological space of Ω is totally disconnected. Finally, suitable examples are constructed, showing how for each Ω, such that the above-mentioned quotient is not totally disconnected, the set S and the function f can be chosen so that the distance from f to $C(\Omega, S)$ be strictly larger than the $C(\Omega)$-norm of d_f^S.

MSC: primary 46E15; 46E40; 41A65

Keywords: Banach spaces of vector-valued continuous functions; Distance formula

E-mail address: burlando@dima.unige.it.

0021-9045/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2005.07.003

1. Introduction

Throughout this paper, when the scalar field is not specified, we assume that it may be either $\mathbb{C}$ or $\mathbb{R}$ and denote it by $\mathbb{K}$.

For each normed space $\mathcal{X}$, let $\| \ \|_{\mathcal{X}}$ and $0_{\mathcal{X}}$ denote, respectively, the norm and the zero element of $\mathcal{X}$. Also, for each $x \in \mathcal{X}$ and each $\varepsilon > 0$, let $B_{\mathcal{X}}(x, \varepsilon)$ stand for the open ball in $\mathcal{X}$ which is centered at x and has a radius equal to ε. Finally, if $\mathcal{S}$ is a nonempty subset of $\mathcal{X}$, for each $\zeta \in \mathcal{X}$ let $d_{\mathcal{X}}(\zeta, \mathcal{S})$ denote the distance from ζ to $\mathcal{S}$ in $\mathcal{X}$, namely,

$$d_{\mathcal{X}}(\zeta, \mathcal{S}) = \inf\left\{\|\zeta - s\|_{\mathcal{X}} : s \in \mathcal{S}\right\}.$$

In a forthcoming paper [Bur], we have provided a formula for the distance from $f \in L_p(\mu, X)$ (where X is a Banach space, $(\mathfrak{X}, \mathfrak{M}, \mu)$ is a positive measure space and $1 \leqslant p \leqslant \infty$) to the set $L_p(\mu, S)$ of all elements of $L_p(\mu, X)$ whose range is (μ-essentially) contained in a given nonempty subset S of X. Indeed, in [Bur] we have proved that, if S is such that $L_p(\mu, S) \neq \emptyset$, then the equivalence class d_f^S of scalar-valued functions on $\mathfrak{X}$, defined by $\mathrm{d}_f^S(t) = d_X\big(f(t), S\big)$ for μ-a.e. $t \in \mathfrak{X}$, belongs to $L_p(\mu)$ and

$$d_{L_p(\mu,X)}\big(f, L_p(\mu, S)\big) = \|\mathrm{d}_f^S\|_{L_p(\mu)}$$

(see [Bur, Proposition 3.8 and Theorem 3.11]; this generalizes the formulae provided in [LC, 2.10] and in [L, Theorem 5], dealing with the special case in which S is a vector subspace of X).

We are concerned here with the corresponding approximation problem in the Banach space $C(\Omega, X)$, for a compact topological space Ω and a Banach space X.

Given a nonempty subset S of X, one could wonder whether the distance from $f \in C(\Omega, X)$ to the set $C(\Omega, S)$ of all $g \in C(\Omega, X)$ with $g(\Omega) \subset S$ is equal to the norm in $C(\Omega)$ of the function

$$\mathrm{d}_f^S : \Omega \ni t \longmapsto d_X\big(f(t), S\big) \in \mathbb{K}.$$

An elementary example (with $\Omega = \overline{B_{\mathbb{K}}(0, 1)}$, $X = \mathbb{K}$ and $S = \mathbb{K} \setminus \{0\}$) shows that it is not so. Indeed, if we consider the continuous function $\imath : \overline{B_{\mathbb{K}}(0, 1)} \ni t \longmapsto t \in \mathbb{K}$, we have

$$\mathrm{d}_\imath^{\mathbb{K}\setminus\{0\}}(t) = d_{\mathbb{K}}(t, \mathbb{K} \setminus \{0\}) = 0 \quad \text{for all } t \in \overline{B_{\mathbb{K}}(0, 1)},$$

and consequently $\left\|\mathrm{d}_\imath^{\mathbb{K}\setminus\{0\}}\right\|_{C(\overline{B_{\mathbb{K}}(0,1)})} = 0$. Furthermore, it is not difficult to verify that for each $g \in C\big(\overline{B_{\mathbb{K}}(0, 1)}\big)$ satisfying $\|g - \imath\|_{C(\overline{B_{\mathbb{K}}(0,1)})} \leqslant 1$, we have $0 \in g\big(\overline{B_{\mathbb{K}}(0, 1)}\big)$: this follows from the intermediate value theorem for $\mathbb{K} = \mathbb{R}$; for $\mathbb{K} = \mathbb{C}$, by the Brouwer fixed-point theorem (see for instance [DS, p. 468]) there exists $x_0 \in \overline{B_{\mathbb{K}}(0, 1)}$ such that $x_0 = \imath(x_0) - g(x_0) = x_0 - g(x_0)$, and consequently $g(x_0) = 0$. Since the distance from $\imath$ to the constant nonzero functions is easily seen to be equal to 1, we conclude that

$$d_{C(\overline{B_{\mathbb{K}}(0,1)})}\Big(\imath, C\big(\overline{B_{\mathbb{K}}(0, 1)}, \mathbb{K} \setminus \{0\}\big)\Big) = 1 > 0 = \left\|\mathrm{d}_\imath^{\mathbb{K}\setminus\{0\}}\right\|_{C(\overline{B_{\mathbb{K}}(0,1)})}.$$

It is known, however, that the desired equality holds if S is a vector subspace of the Banach space X. Indeed, this is proved in Lemma 2 of [FC] under the hypothesis that the compact topological space Ω be Hausdorff, and—as remarked in [LC, p. 134, notes on Chapter 2]—can be derived from the formula provided in [Buc, Theorem 2], in which the distance from $f \in C(\Omega, E)$ (where E is a normed space and Ω is a compact topological space) to a given $C(\Omega)$-submodule of $C(\Omega, E)$ is computed. We recall that another result generalizing [FC, Lemma 2] is [LGC, 2.5], in which the distance from $f \in C(K, E)$ (where K is a compact Hausdorff space and E is a normed space) to the subspace $C(K, H)$—with respect to the norm on $C(K, E)$ which is induced by any monotone norm α on $C(K)$—is proved to be equal to the α-norm of d_f^H for every vector subspace H of E. We also recall that in [JMN, 2.3] the distance from an E-valued (where E is a nonzero normed space) bounded continuous function f on a topological space T to the set of all bounded continuous functions from T into E with values in $E \setminus \{0_E\}$ is proved to coincide with the infimum of all $\delta > 0$ for which the continuous function $t \longmapsto f(t)/\|f(t)\|_E$, defined on the set of all $t \in T$ such that $\|f(t)\|_E \geqslant \delta$, has a continuous extension from T into $\partial B_E(0_E, 1)$.

In this paper we give sufficient conditions (on the compact topological space Ω or on the subset S of the Banach space X) in order that equality $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) = \|\mathrm{d}_f^S\|_{C(\Omega)}$ be satisfied for every $f \in C(\Omega, X)$.

In Section 2, we gather some preliminaries, which we will need in the continuation of the paper and which mainly concern the properties of the distance function $d_X(., S)$ and of d_f^S, as well as a suitable equivalence relation $\sim_\Omega$ on a compact topological space Ω, such that the quotient topological space $\Omega/{\sim_\Omega}$ is Hausdorff. In Section 3 we deal with the problem of finding the distance in $C(\Omega, X)$ from a function f to $C(\Omega, S)$. After observing that the scalar-valued function d_f^S is continuous in Ω for each nonempty subset S of X and each $f \in C(\Omega, X)$ (Proposition 3.4), in Theorem 3.11 we prove that the distance from $f \in C(\Omega, X)$ to the set $C(\Omega, S)$ is not less than the norm of d_f^S in $C(\Omega)$; furthermore, equality holds if the quotient topological space $\Omega/{\sim_\Omega}$ is totally disconnected or S is convex. In Example 3.13 we show that if $\Omega/{\sim_\Omega}$ is not totally disconnected and the dimension of X with respect to the real field is not less than two, a closed connected subset S of X and a continuous X-valued function f on Ω can be chosen so that the norm of d_f^S in $C(\Omega)$ be strictly less than $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big)$. Finally, Example 3.14 shows that if X is a one-dimensional real normed space, then equality $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) = \|\mathrm{d}_f^S\|_{C(\Omega)}$ holds for every $f \in C(\Omega, X)$ if and only if $\Omega/{\sim_\Omega}$ is totally disconnected or S is convex.

2. Preliminaries

Let $\mathcal{X}$ be a Banach space and $\mathcal{S}$ be a nonempty subset of $\mathcal{X}$. It is easily seen that

$$d_{\mathcal{X}}(x, \mathcal{S}) = d_{\mathcal{X}}\big(x, \overline{\mathcal{S}}\big) \quad \text{for all } x \in \mathcal{X}. \tag{2.1}$$

We also recall that the function

$$d_{\mathcal{X}}(., \mathcal{S}) : \mathcal{X} \ni x \longmapsto d_{\mathcal{X}}(x, \mathcal{S}) \in \mathbb{R}$$

is continuous, as

$$|d_{\mathcal{X}}(x, \mathcal{S}) - d_{\mathcal{X}}(y, \mathcal{S})| \leqslant \|x - y\|_{\mathcal{X}} \quad \text{for all } x, y \in \mathcal{X}. \tag{2.2}$$

Now let $\mathcal{Y}$ be a closed subspace of $\mathcal{X}$, and let $\pi_{\mathcal{Y}} : \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{Y}$ denote the canonical quotient map. We remark that

$$d_{\mathcal{X}}(x, \mathcal{Y}) = \|\pi_{\mathcal{Y}} x\|_{\mathcal{X}/\mathcal{Y}} \leqslant \|x\|_{\mathcal{X}} \quad \text{for every } x \in \mathcal{X}. \tag{2.3}$$

Let $\mathcal{D}$ be a set and X be a Banach space.

If S is a nonempty subset of X, following [Bur, Definition 3.6] we define, for each function $f : \mathcal{D} \to X$, the scalar-valued function $\mathrm{d}_f^S : \mathcal{D} \to \mathbb{K}$ as follows:

$$\mathrm{d}_f^S(t) = d_X\big(f(t), S\big) \quad \text{for all } t \in \mathcal{D}.$$

Like in [Bur, (3.2)], from (2.2) we obtain

$$0 \leqslant \mathrm{d}_f^S(t) \leqslant \|f(t)\|_X + d_X(0_X, S) \quad \text{for all } t \in \mathcal{D}. \tag{2.4}$$

Furthermore, if f and g are X-valued functions, defined on $\mathcal{D}$, from (2.2) it follows that (like in [Bur, (3.5)])

$$|\mathrm{d}_f^S(t) - \mathrm{d}_g^S(t)| \leqslant \|f(t) - g(t)\|_X \quad \text{for all } t \in \mathcal{D}. \tag{2.5}$$

Finally, for each closed subspace Y of X and each function $f : \mathcal{D} \to X$, from (2.3) we obtain

$$\mathrm{d}_f^Y(t) = \|(\pi_Y \circ f)(t)\|_{X/Y} \leqslant \|f(t)\|_X \quad \text{for all } t \in \mathcal{D} \tag{2.6}$$

(which is [Bur, (3.3)]).

If Ω is a compact topological space and X is a Banach space, let $C(\Omega, X)$ denote the Banach space of all continuous functions from Ω into X. We recall that

$$\|f\|_{C(\Omega,X)} = \max\{\|f(t)\|_X : t \in \Omega\} \quad \text{for every } f \in C(\Omega, X).$$

We write $C(\Omega)$ for $C(\Omega, \mathbb{K})$.

For each compact topological space Ω, let $\sim_\Omega$ denote the equivalence relation on Ω defined by

$$s \sim_\Omega t \quad \text{if } f(s) = f(t) \text{ for every continuous function } f : \Omega \to \mathbb{R}.$$

It is clear that

$$s,\ t \in \Omega, \quad s \sim_\Omega t \implies f(s) = f(t) \quad \text{for every continuous function } f : \Omega \to \mathbb{C}. \tag{2.7}$$

We denote by q_Ω the canonical quotient map from Ω onto the quotient topological space $\Omega/{\sim_\Omega}$. Also, for each Banach space X, let $Q_\Omega^X : C(\Omega/{\sim_\Omega}, X) \to C(\Omega, X)$ be the linear map defined by

$$Q_\Omega^X u = u \circ q_\Omega \quad \text{for every } u \in C(\Omega/{\sim_\Omega}, X).$$

Notice that Q_Ω^X is an isometry, as q_Ω is onto.

Since Ω is compact, the quotient topology of $\Omega/{\sim_\Omega}$ coincides with the weak topology induced by the family $\mathfrak{C}$ of all functions $u : \Omega/{\sim_\Omega} \to \mathbb{R}$ such that $u \circ q_\Omega$ is continuous on

Ω (namely, see [GJ, 3.3], with the smallest topology on $\Omega/{\sim_\Omega}$ with respect to which every element of $\mathfrak{C}$ is continuous). Indeed, the latter topology is Hausdorff (as established in the proof of [GJ, Theorem 3.9]) and is weaker than or equal to the quotient topology (with respect to which every element of $\mathfrak{C}$ is easily seen to be continuous). Furthermore, $\Omega/{\sim_\Omega}$ is compact under the quotient topology (as Ω is compact), and it is well known that any one-to-one continuous map from a compact topological space onto a Hausdorff one is a homeomorphism. Then the following result can be derived from [GJ, Theorem 3.9] (notice that Theorem 3.9 of [GJ] deals only with real-valued continuous functions; however, once the surjectivity of $Q_\Omega^{\mathbb{R}}$ is proved, the surjectivity of $Q_\Omega^{\mathbb{C}}$ is a straightforward consequence of it).

Theorem 2.8 (*see Gillman and Jerison [GJ, Theorem 3.9]*). *For each compact topological space Ω, the quotient topological space $\Omega/{\sim_\Omega}$ is compact and Hausdorff. Furthermore, $Q_\Omega^{\mathbb{K}}$ is onto. Hence, the Banach spaces $C(\Omega/{\sim_\Omega})$ and $C(\Omega)$ are isometrically isomorphic.*

The following is a straightforward consequence of Urysohn's lemma.

Proposition 2.9. *If Ω is a Hausdorff compact topological space, for every $s, t \in \Omega$ we have*

$$s \sim_\Omega t \iff s = t.$$

Then, under the hypothesis of Proposition 2.9, $\Omega/{\sim_\Omega}$ can be identified with Ω.

Let $\mathbb{Z}_+$ stand for the set of the strictly positive integers.

For each compact topological space Ω and each Banach space X, let $\Pi(\Omega, X)$ denote the vector subspace of $C(\Omega, X)$ which is spanned by the multiples of scalar-valued continuous functions by elements of X, namely, the subspace of all $f \in C(\Omega, X)$ such that there exist $n \in \mathbb{Z}_+$, $\varphi_1, \ldots, \varphi_n \in C(\Omega)$ and $x_1, \ldots, x_n \in X$ satisfying $f(t) = \sum_{k=1}^n \varphi_k(t)x_k$ for all $t \in \Omega$.

The following result will be useful to us in the sequel.

Theorem 2.10 (*Franchetti and Cheney [FC, Lemma 1]; Light and Cheney [LC, proof of 1.13]; Schmets [S, I.5.3]*). *Let Ω be a Hausdorff compact topological space and X be a Banach space. Then $\Pi(\Omega, X)$ is dense in $C(\Omega, X)$.*

3. A distance formula in spaces of vector-valued continuous functions

We begin by extending (in Theorem 3.2 below) the isomorphism result of Theorem 2.8 to vector-valued functions. This extension will enable us to restrict ourselves to the case of a Hausdorff compact topological space, when necessary.

Lemma 3.1. *Let Ω be a compact topological space, $s, t \in \Omega$ and X be a Banach space. Then*

$$s \sim_\Omega t \implies f(s) = f(t) \quad \text{for every } f \in C(\Omega, X).$$

If in addition X is nonzero, then the reverse implication also holds.

Proof. We begin by remarking that, by taking (2.7) into account, we obtain

$$s \sim_\Omega t \iff h(s) = h(t) \quad \text{for every continuous function } f : \Omega \to \mathbb{K}. \tag{3.1.1}$$

Now suppose that $s \sim_\Omega t$ and consider any continuous function $f : \Omega \to X$. Also, let X^* denote the dual space of X. Then from (3.1.1) it follows that

$$\langle f(s), x^* \rangle = \langle f(t), x^* \rangle \quad \text{for every } x^* \in X^*,$$

which gives $f(s) = f(t)$.

Finally, let us assume that $X \neq \{0_X\}$ and $f(s) = f(t)$ for every $f \in C(\Omega, X)$. Fix $x_0 \in X \setminus \{0_X\}$. For each $h \in C(\Omega)$, we have $h x_0 \in C(\Omega, X)$. Then $h(s)x_0 = h(t)x_0$, which, since $x_0 \neq 0_X$, gives $h(s) = h(t)$. Hence $s \sim_\Omega t$. □

Theorem 3.2. *Let Ω be a compact topological space and X be a Banach space. Then the linear isometry*

$$Q_\Omega^X : C(\Omega/{\sim_\Omega}, X) \ni u \longmapsto u \circ q_\Omega \in C(\Omega, X)$$

(*introduced in Section* 2) *is onto. Hence, the Banach spaces $C(\Omega/{\sim_\Omega}, X)$ and $C(\Omega, X)$ are isometrically isomorphic.*

Proof. Let $f \in C(\Omega, X)$. By virtue of Lemma 3.1, there exists a function $u_f : \Omega/{\sim_\Omega} \to X$ such that $u_f\big(q_\Omega(t)\big) = f(t)$ for every $t \in \Omega$. Now let G be an open subset of X. Then $f^{-1}(G)$ is open in Ω. Furthermore, from Lemma 3.1 it also follows that, for each $s, t \in \Omega$ satisfying $s \sim_\Omega t$, we have

$$s \in f^{-1}(G) \iff t \in f^{-1}(G)\,.$$

Hence $u_f^{-1}(G) = q_\Omega\big(f^{-1}(G)\big)$ is open in $\Omega/{\sim_\Omega}$.

We have thus proved that $u_f \in C(\Omega/{\sim_\Omega}, X)$. Since

$$Q_\Omega^X u_f = u_f \circ q_\Omega = f\,,$$

we obtain the desired result. □

We remark that the fact that $C(\Omega/{\sim_\Omega}, X)$ and $C(\Omega, X)$ are isometrically isomorphic could also be derived from Theorem 2.8 by using the injective tensor product (as the completion of the tensor product $C(K) \otimes X$ under the injective norm is isometrically isomorphic to $C(K, X)$ for every compact topological space K—see [DF, 4.2.(2)], and the injective norm satisfies the metric mapping property—see [DF, 4.1.(5)]).

For the reader's convenience, in Corollary 3.3 below we derive the density of $\Pi(\Omega, X)$ in $C(\Omega, X)$ for every compact space Ω (which is claimed to hold, for instance, in the proof of [DF, 4.2.(2)]) from Theorems 3.2 and 2.10.

Corollary 3.3. *Let Ω be a compact topological space and X be a Banach space. Then $\Pi(\Omega, X)$ is dense in $C(\Omega, X)$.*

Proof. From Theorems 2.8 and 2.10 it follows that $\Pi(\Omega/{\sim_\Omega}, X)$ is dense in $C(\Omega/{\sim_\Omega}, X)$. Since Q_Ω^X is onto by Theorem 3.2, we conclude that $Q_\Omega^X\big(\Pi(\Omega/{\sim_\Omega}, X)\big)$ is dense in $C(\Omega, X)$. Now, in order to obtain the desired result, it suffices to prove that $Q_\Omega^X\big(\Pi(\Omega/{\sim_\Omega}, X)\big) \subset \Pi(\Omega, X)$.

Let $u \in \Pi(\Omega/{\sim_\Omega}, X)$. Then there exist $n \in \mathbb{Z}_+, \psi_1, \dots, \psi_n \in C(\Omega/{\sim_\Omega})$ and $x_1, \dots, x_n \in X$ such that

$$u(\tau) = \sum_{k=1}^{+\infty} \psi_k(\tau) x_k \quad \text{for all } \tau \in \Omega/{\sim_\Omega} .$$

This gives

$$(Q_\Omega^X u)(t) = u\big(q_\Omega(t)\big) = \sum_{k=1}^{n} (\psi_k \circ q_\Omega)(t) x_k \quad \text{for all } t \in \Omega.$$

Since $\psi_k \circ q_\Omega \in C(\Omega)$ for all $k = 1, \dots, n$, it follows that $Q_\Omega^X u \in \Pi(\Omega, X)$, which proves the desired inclusion and establishes the corollary. $\square$

We remark that, by using Theorem 2.8, it is not difficult to verify that the opposite inclusion with respect to the one stated in the proof of Corollary 3.3 also holds: namely,

$$Q_\Omega^X\big(\Pi(\Omega/{\sim_\Omega}, X)\big) = \Pi(\Omega, X)$$

for every compact topological space Ω and every Banach space X.

Now we will state some properties of the scalar-valued function d_f^S, for a continuous function f from a compact topological space into a Banach space X and for a nonempty subset S of X.

The following is a consequence of the continuity of $d_X(., S)$ (see (2.2)) and of (2.4).

Proposition 3.4. *Let Ω be a compact topological space, X be a Banach space, S be a nonempty subset of X and $f \in C(\Omega, X)$. Then*

$$\mathrm{d}_f^S \in C(\Omega) \quad \textit{and} \quad \|\mathrm{d}_f^S\|_{C(\Omega)} \leqslant \|f\|_{C(\Omega,X)} + d_X(0_X, S).$$

From Proposition 3.4 and from (2.6), we obtain the following result.

Proposition 3.5. *Let Ω be a compact topological space, X be a Banach space and Y be a closed subspace of X. Then, for each $f \in C(\Omega, X)$, we have*

$$\pi_Y \circ f \in C(\Omega, X/Y) \quad \textit{and} \quad \mathrm{d}_f^Y \in C(\Omega).$$

Furthermore,

$$\|\pi_Y \circ f\|_{C(\Omega,X/Y)} = \|\mathrm{d}_f^Y\|_{C(\Omega)} \leqslant \|f\|_{C(\Omega,X)}.$$

From (2.5) and Proposition 3.4 we obtain the following analogue of [Bur, Lemma 3.9] for spaces of continuous functions.

Proposition 3.6. *Let Ω be a compact topological space, X be a Banach space and S be a nonempty subset of X. Then the map*

$$\mathrm{D}_{\Omega}^{S} : C(\Omega, X) \ni f \longmapsto \mathrm{d}_{f}^{S} \in C(\Omega)$$

is 1-*Lipschitz* (*and is consequently continuous*).

Definition 3.7. Let Ω be a compact topological space and X be a Banach space.

For each nonempty subset S of X, we set

$$C(\Omega, S) = \{f \in C(\Omega, X) : f(t) \in S \text{ for all } t \in \Omega\}.$$

For each subset E of a set $\mathcal{D}$, let 1_E denote the *characteristic function* of E. Namely,

$$1_E : \mathcal{D} \ni t \longmapsto \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E \end{cases} \in \mathbb{K}.$$

Notice that $1_{\mathcal{D}}(t) = 1$ and $1_{\emptyset}(t) = 0$ for all $t \in \mathcal{D}$.

We remark that, under the hypothesis of Definition 3.7, we have $C(\Omega, S) \supset \{x\, 1_\Omega : x \in S\}$, and consequently $C(\Omega, S)$ is nonempty.

Remark 3.8. Let Ω be a compact topological space, X be a Banach space and S be a nonempty subset of X. We remark that, for each $u \in C(\Omega/\sim_\Omega, X)$ and for each $t \in \Omega$, we have

$$\mathrm{d}_u^S\big(q_\Omega(t)\big) = d_X\Big(u\big(q_\Omega(t)\big), S\Big) = \mathrm{d}_{u \circ q_\Omega}^S(t).$$

Hence

$$\mathrm{d}_u^S \circ q_\Omega = \mathrm{d}_{u \circ q_\Omega}^S \quad \text{or, equivalently,} \quad Q_\Omega^{\mathbb{K}} \mathrm{d}_u^S = \mathrm{d}_{Q_\Omega^X u}^S$$
$$\text{for every } u \in C(\Omega/\sim_\Omega, X). \tag{3.8.1}$$

Since $Q_\Omega^{\mathbb{K}}$ is an isometry, from (3.8.1) it follows that

$$\|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)} = \|\mathrm{d}_{Q_\Omega^X u}^S\|_{C(\Omega)} = \|\mathrm{d}_{u \circ q_\Omega}^S\|_{C(\Omega)} \quad \text{for every } u \in C(\Omega/\sim_\Omega, X). \tag{3.8.2}$$

Finally, notice that from (3.8.1) we also obtain

$$Q_\Omega^{\mathbb{K}}\big(\mathrm{D}_{\Omega/\sim_\Omega}^S(u)\big) = Q_\Omega^{\mathbb{K}} \mathrm{d}_u^S = \mathrm{d}_{Q_\Omega^X u}^S = \mathrm{D}_\Omega^S\big(Q_\Omega^X u\big) \quad \text{for each } u \in C(\Omega/\sim_\Omega, X),$$

namely,

$$Q_\Omega^{\mathbb{K}} \circ \mathrm{D}_{\Omega/\sim_\Omega}^S = \mathrm{D}_\Omega^S \circ Q_\Omega^X.$$

We call a function $g : \mathcal{D} \to \mathcal{E}$ (where $\mathcal{D}$ and $\mathcal{E}$ are sets) *simple* if its range $g(\mathcal{D})$ is finite.

If Ω is a compact topological space and X is a Banach space, let $C\Sigma(\Omega, X)$ denote the vector subspace of $C(\Omega, X)$ consisting of all the X-valued continuous simple functions on Ω. Notice that $C\Sigma(\Omega, X)$ contains all the X-valued constant functions on Ω, and consists of them exactly if Ω is connected.

Proposition 3.9. *Let Ω be a compact topological space, X be a Banach space and S be a nonempty subset of X. Then*:

$$(Q_\Omega^X u)(\Omega) = u(\Omega/\sim_\Omega) \quad \textit{for every } u \in C(\Omega/\sim_\Omega, X); \tag{3.9.1}$$

$$C(\Omega, S) = Q_\Omega^X\big(C(\Omega/\sim_\Omega, S)\big); \tag{3.9.2}$$

$$C\Sigma(\Omega, X) = Q_\Omega^X\big(C\Sigma(\Omega/\sim_\Omega, X)\big). \tag{3.9.3}$$

Proof. Since

$$Q_\Omega^X u = u \circ q_\Omega \quad \text{for every } u \in C(\Omega/\sim_\Omega, X),$$

we obtain (3.9.1) by the surjectivity of q_Ω. Now (3.9.1), together with Theorem 3.2, yields (3.9.2) and (3.9.3). The proof is thus complete. □

Lemma 3.10. *Let Ω be a compact topological space, X be a Banach space and S be a nonempty subset of X. Then*

$$\begin{aligned} d_{C(\Omega/\sim_\Omega, X)}\big(u, C(\Omega/\sim_\Omega, S)\big) &= d_{C(\Omega,X)}\big(Q_\Omega^X u, C(\Omega, S)\big) \\ &= d_{C(\Omega,X)}\big(u \circ q_\Omega, C(\Omega, S)\big) \end{aligned}$$

for every $u \in C(\Omega/\sim_\Omega, X)$.

Proof. Let $u \in C(\Omega/\sim_\Omega, X)$. Since $u \circ q_\Omega = Q_\Omega^X u$ and Q_Ω^X is a linear isometry, from (3.9.2) we obtain

$$\begin{aligned} & d_{C(\Omega,X)}\big(u \circ q_\Omega, C(\Omega, S)\big) \\ &\quad = d_{C(\Omega,X)}\big(Q_\Omega^X u, C(\Omega, S)\big) \\ &\quad = d_{C(\Omega,X)}\Big(Q_\Omega^X u, Q_\Omega^X\big(C(\Omega/\sim_\Omega, S)\big)\Big) \\ &\quad = \inf\big\{\|Q_\Omega^X u - Q_\Omega^X v\|_{C(\Omega,X)} : v \in C(\Omega/\sim_\Omega, S)\big\} \\ &\quad = \inf\big\{\|u - v\|_{C(\Omega/\sim_\Omega,X)} : v \in C(\Omega/\sim_\Omega, S)\big\} = d_{C(\Omega/\sim_\Omega,X)}\big(u, C(\Omega/\sim_\Omega, S)\big), \end{aligned}$$

which establishes the desired result. □

Theorem 3.11. *Let Ω be a compact topological space, X be a Banach space and S be a nonempty subset of X. Then*:

(3.11.1) $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) \geqslant \|\mathrm{d}_f^S\|_{C(\Omega)}$ for every $f \in C(\Omega, X)$;

(3.11.2) *if, in addition, at least one of the following two conditions is satisfied*:

- (3.11.2.1) *$\Omega/\sim_\Omega$ is totally disconnected*
- (3.11.2.2) *S is convex,*

we have

$$d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) = \|\mathrm{d}_f^S\|_{C(\Omega)} \quad \textit{for every } f \in C(\Omega, X).$$

Proof. In order to obtain (3.11.1), we proceed in a manner analogous to the proof of the corresponding inequality in [Bur, Theorem 3.11]. Indeed, given $f \in C(\Omega, X)$, for each

$g \in C(\Omega, S)$ we have

$$\|f(t) - g(t)\|_X \geqslant d_X\big(f(t), S\big) = \mathrm{d}_f^S(t) \quad \text{for all } t \in \Omega.$$

This gives

$$\|f - g\|_{C(\Omega,X)} \geqslant \|\mathrm{d}_f^S\|_{C(\Omega)},$$

which in turn yields the desired inequality.

Now we prove (3.11.2).

First suppose Ω to be Hausdorff.

Let $f \in C(\Omega, X)$. By virtue of (3.11.1), we have

$$d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) \geqslant \|\mathrm{d}_f^S\|_{C(\Omega)}.$$

We prove that the opposite inequality also holds if at least one of conditions (3.11.2.1) and (3.11.2.2) is satisfied.

Fix $\varepsilon > 0$. We prove that, if condition (3.11.2.1) or condition (3.11.2.2) is satisfied, there exists $f_\varepsilon \in C(\Omega, S)$ such that

$$\|f - f_\varepsilon\|_{C(\Omega,X)} \leqslant \|\mathrm{d}_f^S\|_{C(\Omega)} + \varepsilon. \tag{3.11.3}$$

Since f is continuous, it follows that, for each $t \in \Omega$, there exists an open neighborhood $V_t^{(\varepsilon)}$ of t such that

$$f(V_t^{(\varepsilon)}) \subset B_X\big(f(t), \varepsilon/3\big). \tag{3.11.4}$$

Furthermore, since Ω is compact, there exist a positive integer n_ε and $t_{1,\varepsilon}, \dots, t_{n_\varepsilon,\varepsilon} \in \Omega$ such that

$$\Omega = \bigcup_{k=1}^{n_\varepsilon} V_{t_{k,\varepsilon}}^{(\varepsilon)}. \tag{3.11.5}$$

We remark that from (2.2) and from (3.11.4) we obtain

$$\Big|d_X\big(f(t), S\big) - d_X\big(f(t_{k,\varepsilon}), S\big)\Big| \leqslant \|f(t) - f(t_{k,\varepsilon})\|_X < \varepsilon/3$$
$$\text{for every } k \in \{1, \dots, n_\varepsilon\},\ t \in V_{t_{k,\varepsilon}}^{(\varepsilon)}.$$

Hence

$$d_X\big(f(t_{k,\varepsilon}), S\big) < d_X\big(f(t), S\big) + \varepsilon/3$$
$$\text{for every } k = 1, \dots, n_\varepsilon \text{ and } t \in V_{t_{k,\varepsilon}}^{(\varepsilon)}. \tag{3.11.6}$$

For each $k = 1, \dots, n_\varepsilon$, let $y_{k,\varepsilon} \in S$ be such that

$$\|f(t_{k,\varepsilon}) - y_{k,\varepsilon}\|_X < d_X\big(f(t_{k,\varepsilon}), S\big) + \varepsilon/3.$$

Then from (3.11.4) and (3.11.6) we obtain

$$\begin{aligned}&\|f(t)-y_{k,\varepsilon}\|_X\\ &\quad\leqslant \|f(t)-f(t_{k,\varepsilon})\|_X+\|f(t_{k,\varepsilon})-y_{k,\varepsilon}\|_X\\ &\quad<\varepsilon/3+d_X\big(f(t_{k,\varepsilon}),S\big)+\varepsilon/3<d_X\big(f(t),S\big)+\varepsilon=\mathrm{d}_f^S(t)+\varepsilon\\ &\qquad\text{for all } k=1,\dots,n_\varepsilon \text{ and } t\in V_{t_{k,\varepsilon}}^{(\varepsilon)}.\end{aligned}\tag{3.11.7}$$

If condition (3.11.2.1) is satisfied, then Ω is totally disconnected by virtue of Proposition 2.9. Since Ω is Hausdorff, compact and totally disconnected, from [HY, Theorem 2–15] it follows that every point of Ω has a neighborhood basis consisting of sets which are open and closed at the same time. Therefore, it is not restrictive to assume that the sets $V_t^{(\varepsilon)}$ above, $t\in\Omega$, are closed as well as open. Now set

$$W_{1,\varepsilon}=V_{t_{1,\varepsilon}}^{(\varepsilon)}\quad\text{and}\quad W_{k,\varepsilon}=V_{t_{k,\varepsilon}}^{(\varepsilon)}\Bigg\backslash\left(\bigcup_{j=1}^{k-1}V_{t_{j,\varepsilon}}^{(\varepsilon)}\right)\quad\text{for } 1<k\leqslant n_\varepsilon.$$

Notice that $W_{k,\varepsilon}$ is open and closed in Ω for every $k=1,\dots,n_\varepsilon$. Therefore, $1_{W_{k,\varepsilon}}\in C(\Omega)$ for every $k=1,\dots,n_\varepsilon$. Let $f_\varepsilon\in C(\Omega,X)$ be defined by

$$f_\varepsilon(t)=\sum_{k=1}^{n_\varepsilon}y_{k,\varepsilon}\,1_{W_{k,\varepsilon}}(t)\quad\text{for all } t\in\Omega.$$

Notice that

$$W_{j,\varepsilon}\cap W_{k,\varepsilon}=\emptyset\quad\text{for all } j,\ k\in\{1,\dots,n_\varepsilon\}\text{ satisfying } j\neq k.\tag{3.11.8}$$

Furthermore, by virtue of (3.11.5), we have

$$\bigcup_{k=1}^{n_\varepsilon}W_{k,\varepsilon}=\bigcup_{k=1}^{n_\varepsilon}V_{t_{k,\varepsilon}}^{(\varepsilon)}=\Omega.\tag{3.11.9}$$

From (3.11.8) and (3.11.9) it follows that

$$f_\varepsilon(\Omega)\subset\{y_{1,\varepsilon},\dots,y_{n_\varepsilon,\varepsilon}\}\subset S,$$

and consequently $f_\varepsilon\in C(\Omega,S)$. Furthermore, since

$$W_{k,\varepsilon}\subset V_{t_{k,\varepsilon}}^{(\varepsilon)}\quad\text{for every } k=1,\dots,n_\varepsilon,$$

from (3.11.7) and (3.11.8) it follows that, for each $k=1,\dots,n_\varepsilon$ and each $t\in W_{k,\varepsilon}$, we have

$$\|f(t)-f_\varepsilon(t)\|_X=\|f(t)-y_{k,\varepsilon}\|_X<\mathrm{d}_f^S(t)+\varepsilon.$$

Now, by virtue of (3.11.9), we conclude that

$$\begin{aligned}\|f-f_\varepsilon\|_{C(\Omega,X)}&=\max\big\{\max\{\|f(t)-f_\varepsilon(t)\|_X : t\in W_{k,\varepsilon}\} : k=1,\dots,n_\varepsilon\big\}\\ &<\|\mathrm{d}_f^S\|_{C(\Omega)}+\varepsilon,\end{aligned}$$

which gives (3.11.3).

Now assume condition (3.11.2.2) to be satisfied. We proceed in a manner similar to the proof of [S, I.5.3], proceeding toward a finite partition of unity. Indeed, since Ω is compact and Hausdorff, and $V^{(\varepsilon)}_{t_{k,\varepsilon}}$ is open in Ω for all $k = 1, \dots, n_\varepsilon$, from (3.11.5) and from a consequence of Urysohn's lemma (see [R, 2.13]) it follows that there exist $\varphi_{1,\varepsilon}, \dots, \varphi_{n_\varepsilon,\varepsilon} \in C(\Omega)$ such that $\varphi_{k,\varepsilon}(\Omega) \subset [0, 1]$ and the support of $\varphi_{k,\varepsilon}$ is contained in $V^{(\varepsilon)}_{t_{k,\varepsilon}}$ for every $k = 1, \dots, n_\varepsilon$, and, moreover, $\sum_{k=1}^{n_\varepsilon} \varphi_{k,\varepsilon}(t) = 1$ for every $t \in \Omega$. Now let $f_\varepsilon \in C(\Omega, X)$ be defined by

$$f_\varepsilon(t) = \sum_{k=1}^{n_\varepsilon} y_{k,\varepsilon}\, \varphi_{k,\varepsilon}(t) \quad \text{for all } t \in \Omega.$$

Since $y_{k,\varepsilon} \in S$ for every $k = 1, \dots, n_\varepsilon$ and S is convex, it follows that $f_\varepsilon \in C(\Omega, S)$.

From (3.11.7) it follows that, for each $t \in \Omega$, we have

$$\begin{cases} \|f(t) - y_{k,\varepsilon}\|_X < \mathrm{d}^S_f(t) + \varepsilon & \text{for all } k = 1, \dots, n_\varepsilon \text{ such that } t \in V^{(\varepsilon)}_{t_{k,\varepsilon}}, \\ \varphi_{k,\varepsilon}(t) = 0 & \text{for all } k = 1, \dots, n_\varepsilon \text{ such that } t \notin V^{(\varepsilon)}_{t_{k,\varepsilon}}, \end{cases}$$

which gives

$$\varphi_{k,\varepsilon}(t)\, \|f(t) - y_{k,\varepsilon}\|_X \leqslant \varphi_{k,\varepsilon}(t)\big(\mathrm{d}^S_f(t) + \varepsilon\big) \quad \text{for all } k = 1, \dots, n_\varepsilon. \tag{3.11.10}$$

From (3.11.10) it follows that

$$\begin{aligned} \|f(t) - f_\varepsilon(t)\|_X &= \left\| \sum_{k=1}^{n_\varepsilon} \big(f(t) - y_{k,\varepsilon}\big)\varphi_{k,\varepsilon}(t) \right\|_X \leqslant \sum_{k=1}^{n_\varepsilon} \varphi_{k,\varepsilon}(t)\|f(t) - y_{k,\varepsilon}\|_X \\ &\leqslant \sum_{k=1}^{n_\varepsilon} \varphi_{k,\varepsilon}(t)\big(\mathrm{d}^S_f(t) + \varepsilon\big) = \mathrm{d}^S_f(t) + \varepsilon \quad \text{for all } t \in \Omega, \end{aligned}$$

which gives (3.11.3).

We have now proved that, if at least one of conditions (3.11.2.1) and (3.11.2.2) is satisfied, for each $\varepsilon > 0$ there exists $f_\varepsilon \in C(\Omega, S)$ such that (3.11.3) holds, from which we conclude that

$$d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) \leqslant \|\mathrm{d}^S_f\|_{C(\Omega)}.$$

Assertion (3.11.2) is thus established in the special case of a Hausdorff compact topological space. We now turn to the general case of a (possibly non-Hausdorff) compact topological space Ω.

Suppose that at least one of conditions (3.11.2.1) and (3.11.2.2) is satisfied and let $f \in C(\Omega, X)$. By virtue of Theorem 3.2, the linear isometry $Q^X_\Omega : C(\Omega/{\sim_\Omega}, X) \to C(\Omega, X)$ is onto, and consequently there exists $v_f \in C(\Omega/{\sim_\Omega}, X)$ such that $f = Q^X_\Omega v_f$. Since $\Omega/{\sim_\Omega}$ is Hausdorff by Theorem 2.8 (which, by Proposition 2.9, implies that the quotient topological space $(\Omega/{\sim_\Omega})\big/{\sim_{(\Omega/\sim_\Omega)}}$ can be identified with $\Omega/{\sim_\Omega}$), what we have proved above can be applied to compute the distance from v_f to $C(\Omega/{\sim_\Omega}, S)$, yielding

$$d_{C(\Omega/\sim_\Omega,X)}\big(v_f, C(\Omega/{\sim_\Omega}, S)\big) = \|\mathrm{d}^S_{v_f}\|_{C(\Omega/\sim_\Omega)}. \tag{3.11.11}$$

Since Q_Ω^X is a linear isometry, from Lemma 3.10, (3.11.11) and (3.8.2) we obtain

$$d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) = d_{C(\Omega,X)}\big(Q_\Omega^X v_f, C(\Omega, S)\big) = d_{C(\Omega/\sim_\Omega,X)}\big(v_f, C(\Omega/\sim_\Omega, S)\big)$$
$$= \|\mathrm{d}^S_{v_f}\|_{C(\Omega/\sim_\Omega)} = \|\mathrm{d}^S_{Q_\Omega^X v_f}\|_{C(\Omega)} = \|\mathrm{d}^S_f\|_{C(\Omega)},$$

which establishes (3.11.2) in the general case and concludes the proof. □

We observe that the function f_ε satisfying (3.11.3), constructed in the proof of Theorem 3.11 in the case in which Ω is Hausdorff and condition (3.11.2.1) is satisfied (that is, Ω is a totally disconnected Hausdorff compact topological space), is simple as well as continuous. Since $\mathrm{d}^X_f = 0_{C(\Omega)}$, for $S = X$ (3.11.3) gives

$$\|f - f_\varepsilon\|_{C(\Omega,X)} \leqslant \varepsilon.$$

This enables us to conclude that $C\Sigma(\Omega, X)$ is dense in $C(\Omega, X)$ if Ω is as above.

Now assume Ω to be a (possibly non-Hausdorff) compact topological space, such that $\Omega/\sim_\Omega$ is totally disconnected. Since $\Omega/\sim_\Omega$ is compact and Hausdorff (see Theorem 2.8), from what we have remarked above it follows that $C\Sigma(\Omega/\sim_\Omega, X)$ is dense in $C(\Omega/\sim_\Omega, X)$. Then, by applying (3.9.3) and Theorem 3.2, we obtain the following result.

Proposition 3.12. *Let Ω be a compact topological space, such that $\Omega/\sim_\Omega$ is totally disconnected. Then $C\Sigma(\Omega, X)$ is dense in $C(\Omega, X)$ for every Banach space X.*

If $\Omega/\sim_\Omega$ fails to be totally disconnected and S fails to be convex, the inequality in (3.11.1) may be strict (even if S is connected and closed), as Example 3.13 below shows. Indeed, in this example we will prove that, for each compact topological space Ω for which $\Omega/\sim_\Omega$ is not totally disconnected and for each Banach space X whose dimension with respect to the real field is not less than two, there exist a nonempty connected closed subset S of X and $f \in C(\Omega, X)$ such that $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) > \|\mathrm{d}^S_f\|_{C(\Omega)}$.

Example 3.13. Let Ω be a compact topological space, such that $\Omega/\sim_\Omega$ is not totally disconnected, and let X be a Banach space (over $\mathbb{K}$) of dimension not less than two with respect to the real field (that is, X is either a nonzero complex Banach space or a real Banach space of dimension not less than two).

We prove that there exist a nonempty connected closed subset S of X and $f \in C(\Omega, X)$ such that $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) \geqslant 2$ and $\|\mathrm{d}^S_f\|_{C(\Omega)} = 1$.

Let $\mathcal{C}$ be a connected component of $\Omega/\sim_\Omega$, containing more than one point. Since $\Omega/\sim_\Omega$ is compact and Hausdorff (see Theorem 2.8), so is $\mathcal{C}$. Hence, by Urysohn's lemma and by the connectedness of $\mathcal{C}$, there exists $\varphi_0 \in C(\mathcal{C})$ such that $\varphi_0(\mathcal{C}) = [0, 1]$. Now from the Tietze extension theorem (see [HY, 2–31]) it follows that there exists $\varphi \in C(\Omega/\sim_\Omega)$ such that $\varphi_{|\mathcal{C}} = \varphi_0$ and $\varphi(\Omega/\sim_\Omega) = [0, 1]$.

Since X has a dimension larger than or equal to 2 with respect to the real field, from the Riesz lemma and from the compactness of the unit sphere in finite-dimensional normed spaces we conclude that there exist two norm-one vectors $e_1, e_2 \in X$ such that the distance from e_1 to the real subspace of X spanned by e_2 is equal to 1. Hence

$$\|\alpha e_1 + \beta e_2\|_X \geqslant |\alpha| \quad \text{for all } \alpha, \beta \in \mathbb{R}. \tag{3.13.1}$$

Since e_1 and e_2 are linearly independent with respect to the real field, and consequently the linear (with respect to the real field) map $\mathbb{R}^2 \ni (\alpha, \beta) \longmapsto \alpha e_1 + \beta e_2 \in Y$ (where Y denotes the real subspace of X spanned by e_1 and e_2) is a homeomorphism, there exists $\delta > 0$ such that

$$\|\alpha e_1 + \beta e_2\|_X \geqslant \delta\sqrt{\alpha^2 + \beta^2} \quad \text{for all } \alpha,\ \beta \in \mathbb{R}. \tag{3.13.2}$$

We set

$$S_1 = \left\{ s e_2 - e_1 : s \in \left[0, \frac{2}{\delta}\right] \right\}, \quad S_2 = \left\{ s e_2 + e_1 : s \in \left[0, \frac{2}{\delta}\right] \right\},$$
$$S_3 = \left\{ s e_1 + \frac{2}{\delta}\, e_2 : s \in [-1, 1] \right\}$$

and

$$S = S_1 \cup S_2 \cup S_3.$$

Notice that S is (arcwise) connected, but is not convex. Also, S is clearly closed.

Now let $u \in C(\Omega/{\sim_\Omega}, X)$ be defined by

$$u(t) = \big(2\varphi(t) - 1\big)e_1 \quad \text{for all } t \in \Omega/{\sim_\Omega}.$$

From (3.13.1) it follows that, for each $t \in \Omega/{\sim_\Omega}$ and each $s \in [0, 2/\delta]$, we have

$$\|u(t) - (s e_2 - e_1)\|_X = \|2\varphi(t)e_1 - s e_2\|_X \geqslant 2\varphi(t) = \|2\varphi(t)e_1\|_X = \|u(t) + e_1\|_X$$

and

$$\begin{aligned}\|u(t) - (s e_2 + e_1)\|_X &= \big\|2\big(\varphi(t) - 1\big)e_1 - s e_2\big\|_X \\ &\geqslant 2\big(1 - \varphi(t)\big) = \big\|2\big(\varphi(t) - 1\big)e_1\big\|_X = \|u(t) - e_1\|_X.\end{aligned}$$

Since $-e_1 \in S_1$ and $e_1 \in S_2$, we conclude that

$$d_X\big(u(t), S_1\big) = \|u(t) + e_1\|_X = 2\varphi(t)$$

and

$$d_X\big(u(t), S_2\big) = \|u(t) - e_1\|_X = 2\big(1 - \varphi(t)\big) \quad \text{for every } t \in \Omega/{\sim_\Omega}. \tag{3.13.3}$$

From (3.13.2) it follows that, for each $t \in \Omega/{\sim_\Omega}$ and each $s \in [-1, 1]$, we have

$$\begin{aligned}\left\| u(t) - \left(s e_1 + \frac{2}{\delta}\, e_2 \right) \right\|_X &= \left\| \big(2\varphi(t) - s - 1\big)e_1 - \frac{2}{\delta}\, e_2 \right\|_X \\ &\geqslant \delta \sqrt{\big(2\varphi(t) - s - 1\big)^2 + \frac{4}{\delta^2}} \geqslant \delta \left(\frac{2}{\delta}\right) = 2.\end{aligned}$$

Hence

$$d_X\big(u(t), S_3\big) \geqslant 2 \quad \text{for every } t \in \Omega/{\sim_\Omega}\ . \tag{3.13.4}$$

Since $\varphi(t) \in [0, 1]$—and consequently $2\varphi(t), 2\big(1 - \varphi(t)\big) \in [0, 2]$—for every $t \in \Omega/{\sim_\Omega}$, from (3.13.3) and (3.13.4) it follows that

$$\begin{aligned} \mathrm{d}_u^S(t) &= d_X\big(u(t), S\big) = d_X\big(u(t), S_1\big) \wedge d_X\big(u(t), S_2\big) \\ &= 2\Big(\varphi(t) \wedge \big(1 - \varphi(t)\big)\Big) = \begin{cases} 2\varphi(t) & \text{if } t \in \varphi^{-1}([0, 1/2]), \\ 2\big(1 - \varphi(t)\big) & \text{if } t \in \varphi^{-1}([1/2, 1]), \end{cases} \end{aligned}$$

which is less than or equal to 1 for all $t \in \Omega/{\sim_\Omega}$. Since $\varphi(\Omega/{\sim_\Omega}) = [0, 1] \ni 1/2$, we conclude that

$$\|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)} = 1. \tag{3.13.5}$$

Now let $v \in C(\Omega/{\sim_\Omega}, S)$. Suppose first that $v(\mathcal{C}) \cap S_3 \neq \emptyset$. Then there exists $t_0 \in \mathcal{C}$ such that $v(t_0) \in S_3$. Hence, by virtue of (3.13.4), we have

$$\|u - v\|_{C(\Omega/\sim_\Omega, X)} \geqslant \|u(t_0) - v(t_0)\|_X \geqslant d_X\big(u(t_0), S_3\big) \geqslant 2. \tag{3.13.6}$$

Now suppose $v(\mathcal{C}) \subset S_1 \cup S_2$. Since S_1 and S_2 are closed and disjoint, $\mathcal{C}$ is connected and v is continuous, it follows that there exists $j \in \{1, 2\}$ such that $v(\mathcal{C}) \subset S_j$. Since

$$\varphi(\mathcal{C}) = \varphi_0(\mathcal{C}) = [0, 1],$$

from (3.13.3) we obtain

$$\begin{aligned} &\|u - v\|_{C(\Omega/\sim_\Omega, X)} \\ &\quad = \max\big\{\|u(t) - v(t)\|_X : t \in \Omega/{\sim_\Omega}\big\} \\ &\quad \geqslant \max\big\{\|u(t) - v(t)\|_X : t \in \mathcal{C}\big\} \geqslant \max\big\{d_X\big(u(t), S_j\big) : t \in \mathcal{C}\big\} \\ &\quad = \left.\begin{matrix} \max\{2\varphi(t) : t \in \mathcal{C}\} & \text{if } j = 1 \\ \max\big\{2\big(1 - \varphi(t)\big) : t \in \mathcal{C}\big\} & \text{if } j = 2 \end{matrix}\right\} = 2. \end{aligned} \tag{3.13.7}$$

From (3.13.6) and (3.13.7) we conclude that

$$\|u - v\|_{C(\Omega/\sim_\Omega, X)} \geqslant 2 \quad \text{for every } v \in C(\Omega/{\sim_\Omega}, S),$$

and consequently, by (3.13.5),

$$d_{C(\Omega/\sim_\Omega, X)}\big(u, C(\Omega/{\sim_\Omega}, S)\big) \geqslant 2 > 1 = \|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)}. \tag{3.13.8}$$

Finally, let $f \in C(\Omega, X)$ be defined by $f = u \circ q_\Omega = Q_\Omega^X u$. Then Lemma 3.10, (3.13.8) and (3.8.2) give

$$\begin{aligned} d_{C(\Omega, X)}\big(f, C(\Omega, S)\big) &= d_{C(\Omega/\sim_\Omega, X)}\big(u, C(\Omega/{\sim_\Omega}, S)\big) \geqslant 2 > 1 \\ &= \|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)} = \|\mathrm{d}_f^S\|_{C(\Omega)}, \end{aligned}$$

which establishes the desired result.

The following example deals with the case that is not considered in Example 3.13, that is, the case of continuous functions with values in a one-dimensional real Banach space. Indeed, we will show that, for each compact topological space Ω for which $\Omega/\sim_\Omega$ is not totally disconnected, each real Banach space X of dimension 1 and each disconnected (or, equivalently, non-convex) subset S of X, there exists $f \in C(\Omega, X)$ such that $d_{C(\Omega,X)}\big(f, C(\Omega, S)\big) > \|\mathrm{d}_f^S\|_{C(\Omega)}$.

Example 3.14. Let Ω be a compact topological space, such that $\Omega/\sim_\Omega$ is not totally disconnected, X be a real Banach space of dimension 1 and S be a disconnected subset of X. Without loss of generality, we may assume $X = \mathbb{R}$ (indeed, if we choose a norm-one element e of X, the map $\Lambda : \mathbb{R} \ni \lambda \longmapsto \lambda e \in X$ is an isometric isomorphism, which implies that $d_{C(\Omega)}\big(g, C(\Omega, A)\big) = d_{C(\Omega,X)}\Big(\Lambda \circ g, C\big(\Omega, \Lambda(A)\big)\Big)$ and $\mathrm{d}_g^A = \mathrm{d}_{\Lambda\circ g}^{\Lambda(A)}$ for every continuous function $g : \Omega \to \mathbb{R}$ and every nonempty subset A of $\mathbb{R}$).

We prove that there exists $f \in C(\Omega)$ such that $d_{C(\Omega)}\big(f, C(\Omega, S)\big) > \|\mathrm{d}_f^S\|_{C(\Omega)}$. By virtue of Theorem 3.2, (3.8.2) and Lemma 3.10, and as in Example 3.13, this is equivalent to proving that there exists $u \in C(\Omega/\sim_\Omega)$ such that

$$d_{C(\Omega/\sim_\Omega)}\big(u, C(\Omega/\sim_\Omega, S)\big) > \|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)}. \tag{3.14.1}$$

Let $\mathcal{C}$ and φ be the same as in Example 3.13. We recall (see Example 3.13) that

$$\varphi(\mathcal{C}) = \varphi(\Omega/\sim_\Omega) = [0, 1]. \tag{3.14.2}$$

First assume $\overline{S}$ to be connected. Then $(\inf S, \sup S) \subset \overline{S}$. Since S is disconnected, there exists $\lambda_0 \in (\inf S, \sup S)$ such that $\lambda_0 \notin S$. Notice that λ_0 is an interior point of $\overline{S}$, and consequently there exists $\varepsilon > 0$ such that $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset \overline{S}$. Now let $u \in C(\Omega/\sim_\Omega)$ be defined by

$$u(t) = \lambda_0 + 2\varepsilon\varphi(t) - \varepsilon \quad \text{for all } t \in \Omega/\sim_\Omega .$$

For each $v \in C(\Omega/\sim_\Omega, S)$, $v(\mathcal{C})$ is contained in either $(-\infty, \lambda_0)$ or $(\lambda_0, +\infty)$, as $\lambda_0 \notin S$. On the other hand, from (3.14.2) it follows that

$$u(\mathcal{C}) = u(\Omega/\sim_\Omega) = [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]. \tag{3.14.3}$$

Hence

$$\begin{aligned}&\|u - v\|_{C(\Omega/\sim_\Omega)}\\ &\quad \geqslant \max\{|u(t) - v(t)| : t \in \mathcal{C}\} > \varepsilon \quad \text{for all } v \in C(\Omega/\sim_\Omega, S).\end{aligned} \tag{3.14.4}$$

Since $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \subset \overline{S}$, from (3.14.3) it follows that

$$\mathrm{d}_u^S(t) = d_{\mathbb{R}}\big(u(t), S\big) = 0 \quad \text{for all } t \in \Omega/\sim_\Omega,$$

which, together with (3.14.4), gives

$$d_{C(\Omega/\sim_\Omega)}\big(u, C(\Omega/\sim_\Omega, S)\big) \geqslant \varepsilon > 0 = \|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)},$$

thus establishing (3.14.1).

Now suppose that $\overline{S}$ is not connected. Then there exist $x, y \in \overline{S}$ such that $x < y$ and (x, y) has a nonempty intersection with $\mathbb{R} \setminus \overline{S}$. Hence, the open set $\mathbb{R} \setminus \overline{S}$ has at least one component which is bounded (being contained in (x, y)). Consequently, there exist $a, b \in \overline{S}$ such that $a < b$ and $(a, b) \subset \mathbb{R} \setminus \overline{S}$. Now let $u \in C(\Omega/\sim_\Omega)$ be defined by

$$u(t) = a + \varphi(t)(b - a) \quad \text{for all } t \in \Omega/\sim_\Omega .$$

By virtue of (3.14.2), there exists $t_0 \in \Omega/\sim_\Omega$ such that $\varphi(t_0) = 1/2$ (and consequently $u(t_0) = (a + b)/2$). Since $a, b \in \overline{S}$ and $(a, b) \cap \overline{S} = \emptyset$, from (3.14.2) and (2.1) we derive that

$$\begin{aligned} \mathrm{d}_u^S(t) &= d_{\mathbb{R}}\big(u(t), S\big) = d_{\mathbb{R}}\big(u(t), \overline{S}\big) \\ &= \begin{cases} u(t) - a = \varphi(t)(b - a) & \text{if } t \in \varphi^{-1}([0, 1/2]) \\ b - u(t) = \big(1 - \varphi(t)\big)(b - a) & \text{if } t \in \varphi^{-1}([1/2, 1]) \end{cases} \leqslant \frac{b - a}{2} \\ &= d_{\mathbb{R}}\big(u(t_0), \overline{S}\big) = d_{\mathbb{R}}\big(u(t_0), S\big) = \mathrm{d}_u^S(t_0) \quad \text{for all } t \in \Omega/\sim_\Omega, \end{aligned}$$

which gives

$$\|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)} = \mathrm{d}_u^S(t_0) = \frac{b - a}{2}. \tag{3.14.5}$$

Now let $v \in C(\Omega/\sim_\Omega, S)$. Since $(a, b) \cap S = \emptyset$, it follows that $v(\mathcal{C})$ is contained in either $(-\infty, a]$ or $[b, +\infty)$. On the other hand, from (3.14.2) it follows that $u(\mathcal{C}) = [a, b]$, and consequently

$$\|u - v\|_{C(\Omega/\sim_\Omega)} \geqslant \max\{|u(t) - v(t)| : t \in \mathcal{C}\} \geqslant b - a. \tag{3.14.6}$$

Now (3.14.5) and (3.14.6) give

$$d_{C(\Omega/\sim_\Omega)}\big(u, C(\Omega/\sim_\Omega, S)\big) \geqslant b - a > \frac{b - a}{2} = \|\mathrm{d}_u^S\|_{C(\Omega/\sim_\Omega)},$$

which in turn yields (3.14.1) and completes the example.

The following example shows that Proposition 3.12 does not hold if the hypothesis that $\Omega/\sim_\Omega$ be totally disconnected is dropped. Indeed, we will show that, if $\Omega/\sim_\Omega$ is not totally disconnected and X is nonzero, then $C\Sigma(\Omega, X)$ is not dense in $C(\Omega, X)$.

Example 3.15. Let Ω be a compact topological space, such that $\Omega/\sim_\Omega$ is not totally disconnected, and let X be a nonzero Banach space.

We prove that $C\Sigma(\Omega, X)$ is not dense in $C(\Omega, X)$, which, by virtue of (3.9.3) and of Theorem 3.2, is equivalent to proving that $C\Sigma(\Omega/\sim_\Omega, X)$ is not dense in $C(\Omega/\sim_\Omega, X)$.

Let φ and $\mathcal{C}$ be the same as in Examples 3.13 and 3.14. Since $\varphi(\mathcal{C}) = [0, 1]$ (see Example 3.13), there exist $t_0, t_1 \in \mathcal{C}$ such that $\varphi(t_0) = 0$ and $\varphi(t_1) = 1$. Furthermore, let x_0 be a norm-one element of X, and x_0^* be a norm-one bounded linear functional on X, satisfying $\langle x_0, x_0^* \rangle = 1$.

Now let $u \in C(\Omega/\sim_\Omega, X)$ be defined by

$$u(t) = \varphi(t)x_0 \quad \text{for all } t \in \Omega/\sim_\Omega .$$

For each $v \in C\Sigma(\Omega/\sim_\Omega, X)$, $v(\mathcal{C})$ is a connected subset of $v(\Omega/\sim_\Omega)$, which is finite. Hence, $v(\mathcal{C})$ consists of a single point, that is, there exists $y_v \in X$ such that

$$v(t) = y_v \quad \text{for all } t \in \mathcal{C}.$$

Then

$$\begin{aligned}
&\|u - v\|_{C(\Omega/\sim_\Omega, X)} \\
&\quad \geqslant \max\{\|u(t) - v(t)\|_X : t \in \mathcal{C}\} = \max\{\|\varphi(t)x_0 - y_v\|_X : t \in \mathcal{C}\} \\
&\quad \geqslant \max\{\,|\langle \varphi(t)x_0 - y_v, x_0^*\rangle| : t \in \mathcal{C}\} = \max\{\,|\,\varphi(t) - \langle y_v, x_0^*\rangle| : t \in \mathcal{C}\} \\
&\quad \geqslant \max\{\,|\,\varphi(t_0) - \langle y_v, x_0^*\rangle|\,,\ |\,\varphi(t_1) - \langle y_v, x_0^*\rangle|\,\} \\
&\quad = \max\{\,|\langle y_v, x_0^*\rangle|\,,\ |\,1 - \langle y_v, x_0^*\rangle|\,\} \geqslant \frac{|\langle y_v, x_0^*\rangle| + |\,1 - \langle y_v, x_0^*\rangle|}{2} \geqslant \frac{1}{2}.
\end{aligned}$$

Hence $C\Sigma(\Omega/\sim_\Omega, X)$ is not dense in $C(\Omega/\sim_\Omega, X)$, and the desired result is established.

Finally, we will derive analogues of [Bur, Corollaries 3.12 and 3.13] for Banach spaces of continuous functions, as a consequence of Theorem 3.11.

As a preliminary, we observe that if Y is a closed subspace of a Banach space X and Ω is a compact topological space, then $C(\Omega, Y)$, being a Banach space, is a closed subspace of $C(\Omega, X)$.

Corollary 3.16. *Let X be a Banach space, Y be a closed subspace of X and Ω be a compact topological space. Then*

$$d_{C(\Omega,X)}\big(f, C(\Omega, Y)\big) = \|\pi_{C(\Omega,Y)} f\|_{C(\Omega,X)/C(\Omega,Y)} = \|\pi_Y \circ f\|_{C(\Omega,X/Y)}$$
for every $f \in C(\Omega, X)$.

Proof. Since Y is convex, then (3.11.2) applies, giving

$$d_{C(\Omega,X)}\big(f, C(\Omega, Y)\big) = \|\mathrm{d}_f^Y\|_{C(\Omega)} \quad \text{for every } f \in C(\Omega, X).$$

Now the desired result follows from Proposition 3.5 and from (2.3). □

Notice that Corollary 3.16 is also clearly a consequence of [Buc, Theorem 2] or—for a Hausdorff space Ω—of [FC, Lemma 2].

The following result can be proved proceeding as in [Bur, Corollary 3.13] (by replacing an appeal to density of the countably valued elements of $L_p(\mu, X)$ with an appeal to density of $\Pi(\Omega, X)$ in $C(\Omega, X)$). For the reader's convenience, we will provide an explicit proof for it.

Corollary 3.17. *Let X be a Banach space, Y be a closed subspace of X and Ω be a compact topological space. Then*:

(3.17.1) *the linear map*

$$\Gamma_Y^\Omega : C(\Omega, X) \ni f \longmapsto \pi_Y \circ f \in C(\Omega, X/Y)$$

is bounded and onto;

(3.17.2) *the kernel of* Γ_Y^Ω *coincides with* $C(\Omega, Y)$;
(3.17.3) *the linear map*

$$\Delta_Y^\Omega : C(\Omega, X)/C(\Omega, Y) \ni f + C(\Omega, Y) \longmapsto \pi_Y \circ f \in C(\Omega, X/Y)$$

(*induced by* Γ_Y^Ω *via the first homomorphism theorem*) *is isometric and onto*; *hence, the Banach spaces* $C(\Omega, X)/C(\Omega, Y)$ *and* $C(\Omega, X/Y)$ *are isometrically isomorphic*.

Proof. From Proposition 3.5 we derive that Γ_Y^Ω is well defined and bounded. Furthermore, (3.17.2) follows from Corollary 3.16. Then Δ_Y^Ω is well defined and

$$\Delta_Y^\Omega\big(C(\Omega, X)/C(\Omega, Y)\big) = \Gamma_Y^\Omega\big(C(\Omega, X)\big). \tag{3.17.4}$$

From Corollary 3.16 it also follows that Δ_Y^Ω is an isometry, and consequently has closed range. Then Γ_Y^Ω has closed range by (3.17.4). Now we prove that Γ_Y^Ω is onto.

Let $u \in \Pi(\Omega, X/Y)$. Then there exist $n \in \mathbb{Z}_+$, $\varphi_1, \ldots, \varphi_n \in C(\Omega)$ and $\xi_1, \ldots, \xi_n \in X/Y$ such that

$$u(t) = \sum_{k=1}^{n} \varphi_k(t)\, \xi_k \quad \text{for all } t \in \Omega.$$

For each $k = 1, \ldots, n$, let $x_k \in X$ be such that $\pi_Y x_k = \xi_k$. Now let $f : \Omega \to X$ be the continuous function defined by

$$f(t) = \sum_{k=1}^{n} \varphi_k(t) x_k \quad \text{for all } t \in \Omega.$$

Then

$$(\Gamma_Y^\Omega f)(t) = (\pi_Y \circ f)(t) = \pi_Y f(t) = \sum_{k=1}^{n} \varphi_k(t) \pi_Y x_k = \sum_{k=1}^{n} \varphi_k(t) \xi_k = u(t) \quad \text{for all } t \in \Omega,$$

which gives $\Gamma_Y^\Omega f = u$.

We have thus proved that

$$\Gamma_Y^\Omega\big(C(\Omega, X)\big) \supset \Pi(\Omega, X/Y). \tag{3.17.5}$$

Since $\Gamma_Y^\Omega\big(C(\Omega, X)\big)$ is closed, and $\Pi(\Omega, X/Y)$ is dense in $C(\Omega, X/Y)$ by Theorem 2.10, from (3.17.5) we conclude that Γ_Y^Ω is onto, which completes the proof of (3.17.1). Now from (3.17.4) it follows that Δ_Y^Ω is also onto, which completes the proof of (3.17.3) and establishes the corollary. □

We remark that Corollary 3.17 (and consequently Corollary 3.16) can also be obtained via an injective tensor product: indeed, it can, for instance, be derived from [DF, 4.2.(2), 4.4 (applied to the metric surjection π_Y) and 2.7, Property (4)].

Corollary 3.18. *Let Ω be a compact topological space, X be a Banach space and S be a nonempty subset of X. If at least one of conditions* (3.11.2.1) *and* (3.11.2.2) *is satisfied, then*:
(3.18.1) $d_{C(\Omega,X)}\big(f, C(\Omega, \overline{S})\big) = d_{C(\Omega,X)}\big(f, C(\Omega, S)\big)$ *for every* $f \in C(\Omega, X)$;
(3.18.2) $C(\Omega, \overline{S}) = \overline{C(\Omega, S)}$.

Proof. Since $\mathrm{d}_f^{\overline{S}} = \mathrm{d}_f^S$ by (2.1) (and $\overline{S}$ is convex if S is convex), (3.18.1) follows from Theorem 3.11. Since $C(\Omega, \overline{S})$ is clearly a closed subset of $C(\Omega, X)$, (3.18.1) in turn yields (3.18.2). □

Notice that if none of conditions (3.11.2.1) and (3.11.2.2) is satisfied, then both the inequality $d_{C(\Omega,X)}\big(f, C(\Omega, \overline{S})\big) \leqslant d_{C(\Omega,X)}\big(f, C(\Omega, S)\big)$ and the inclusion $\overline{C(\Omega, S)} \subset C(\Omega, \overline{S})$ may be strict, as the elementary example mentioned in the Introduction shows.

References

[Buc] R.C. Buck, Approximation properties of vector valued functions, Pacific J. Math. 53 (1974) 85–94.
[Bur] L. Burlando, Computing the distance to the functions with range in a given set in Lebesgue spaces of vector-valued functions, to appear in Numer. Funct. Anal. Optim.
[DF] A. Defant, K. Floret, Tensor Norms and Operator Ideals, North-Holland Mathematics Studies, vol. 176, North-Holland, Amsterdam, 1993.
[DS] N. Dunford, J.T. Schwartz, Linear Operators, Part I: General Theory, Interscience Publishers, 1958.
[FC] C. Franchetti, E.W. Cheney, Best approximation problems for multivariate functions, Boll. Un. Mat. Ital. B (5) 18 (1981) 1003–1015.
[GJ] L. Gillman, M. Jerison, Rings of Continuous Functions, D. Van Nostrand, 1960.
[HY] J.G. Hocking, G.S. Young, Topology, Addison-Wesley, Reading, MA, 1961.
[JMN] A. Jimenez-Vargas, J.F. Mena-Jurado, J.C. Navarro-Pascual, Approximation by extreme functions, J. Approx. Theory 97 (1999) 15–30.
[L] W.A. Light, Proximinality in $L_p(S, Y)$, Rocky Mountain J. Math. 19 (1989) 251–259.
[LC] W.A. Light, E.W. Cheney, Approximation Theory in Tensor Product Spaces, Lecture Notes in Mathematics, vol. 1169, Springer, Berlin, 1985.
[LGC] W.A. Light, M. von Golitschek, E.W. Cheney, Approximation with monotone norms in tensor product spaces, J. Approx. Theory 68 (1992) 183–205.
[R] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1970.
[S] J. Schmets, Spaces of Vector-Valued Continuous Functions, Lecture Notes in Mathematics, vol. 1003, Springer, Berlin, 1983.